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BPS operators in $\mathcal{N} = 4$ SYM: Calogero models and 2D fermions

Abhishek Agarwal and Alexios P. Polychronakos

Physics Department, City College of the CUNY
New York, NY 10031, U.S.A.
E-mail: abhishek,alexios@sci.ccny.cuny.edu

Abstract: A connection between the gauge fixed dynamics of protected operators in superconformal Yang-Mills theory in four dimensions and Calogero systems is established. This connection generalizes the free Fermion description of the chiral primary operators of the gauge theory formed out of a single complex scalar to more general operators. In particular, a detailed analysis of protected operators charged under an $su(1|1) \in psu(2,2|4)$ is carried out and a class of operators is identified, whose dynamics is described by the rational super-Calogero model. These results are generalized to arbitrary BPS operators charged under an $su(2|3)$ of the superconformal algebra. Analysis of the non-local symmetries of the super-Calogero model is also carried out, and it is shown that symmetry for a large class of protected operators is a contraction of the corresponding Yangian algebra to a loop algebra.

Keywords: Integrable Equations in Physics, AdS-CFT Correspondence, 1/N Expansion, Matrix Models.

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Contents

1. Introduction

A substantial number of problems related to the study of maximally supersymmetric Yang-Mill theory on $R \times S^3$ can be translated into the study of hamiltonian multi-matrix models. Perhaps the most striking success of this simplification is the successful computation of the spectrum of anomalous dimensions of the gauge theory by mapping the relevant large N matrix models to integrable quantum spin chains [1, 2]. The matrix model in question is nothing but the radial hamiltonian of the gauge theory or the dilatation operator. The dilatation operator is, in general, a complicated multi-matrix model whose Hamiltonian can only be computed order by order in perturbation theory. However if one focuses on protected operators of the gauge theory then the dilatation operator takes on a particularly simple form: indeed it is nothing but a sum of decoupled matrix harmonic oscillators [3, 7, 8]. Though the radial Hamiltonian (when restricted to the sector of protected operators) appears to be a non-interacting system, gauge fixing induces non-trivial interactions among the microscopic degrees of freedom of the Hamiltonian. Gauge fixing is necessary as the radial Hamiltonian inherits a residual $\mathrm{U}(N)$ gauge invariance from the original super Yang-Mills theory. In the simplest case, when one studies the dynamics of BPS operators built out of a single complex scalar even the gauge fixed dynamics turns out to be free: indeed the gauge fixed theory is nothing other than that of a collection of free fermions [4, 9, 3, 7, 8, 10]. However this is not the generic scenario. Investigation of the microscopic dynamics of operators that involve several SYM fields leads, in general, to interacting but integrable particle mechanics that can be understood as generalizations of the celebrated Calogero models.

In this present work, we study this connection between protected operators in $\mathcal{N}$=4 SYM and Calogero models. The underlying motivation is to develop the quantum many-body theories that are relevant for the appropriate generalizations of the free fermion picture of the Hamiltonian description of chiral primaries of the gauge theory formed out of a single complex scalar. In particular, we shall focus on the sub-sector of gauge theory formed out of three complex scalar fields and two fermions known as the $su(2|3)$ sector of $\mathcal{N}$ =4 SYM [11]. The operator content of this subsector is

$$\mathcal{W}_\alpha = \{Z_1, Z_2, Z_3, \Psi_1, \Psi_2\}. \tag{1.1}$$

Z_1, Z_2, Z_3 being three complex chiral scalars and Ψ_1, Ψ_2 being two Fermions. The motivation behind the choice of the $su(2|3)$ sector is its closure under dilatation as all local gauge invariant composite operators formed out of these five fields

$$\mathcal{O} = \prod_m Tr(\mathcal{W}_{\alpha_1} \cdots \mathcal{W}_{\alpha_m}) \tag{1.2}$$

mix only with each other to all orders in perturbation theory [11]. Before elaborating further on the technical details of the dynamics of the protected operators contained in the $su(2|3)$ sector, it is worth laying out a summary of the basic results obtained in the paper. We construct a gauge fixed version of the tree level dilatation operator, which is nothing but a sum of matrix harmonic oscillators and realize it as a generalization of the celebrated supersymmetric Calogero model known as the Euler Calogero model. Not all the excitations of the Euler Calogero model correspond to BPS excitations of the the gauge theory, however, the manifest supersymmetry of the Euler Calogero model can be utilized to pick out those excitations which do correspond to the protected gauge theory operators. This construction has been carried out in section 5 of the paper. The $su(2|3)$ sector also contains a smaller closed subsector of operator mixing namely the so called $su(1|1)$ sector containing a scalar and a single Fermionic field. Within this subsector, we have been able to construct a set of states of the tree level dilatation operator that are protected in the large N limit, but have small anomalous dimensions at finite values of the rank of the gauge group. The gauge theory operators corresponding to these states interpolate between the so called LLM[4] and BMN[5]operators and provide us with a set of non-BPS operators about which one can make non-perturbative/all loops statements by studying the large N limit of the corresponding tree level dilatation operator. For these operators, we are able to recast the gauge fixed large N dilatation operator as the well known rational supersymmetric Calogero model. The construction of the Calogero model and the use of its integrability to completely solve for its dynamics, enumerate the degeneracies of the corresponding gauge theory operators and to study the Yangian symmetry underlying their dynamics has been done in sections 2, 3 and 4.

To recapitulate the free fermion picture of half BPS states it is worth recalling that one can pick any one of the complex scalars, say Z_1 of the $su(2|3)$ sector of the gauge theory and one then has the standard result that all the operators formed out of only Z_1's are half

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BPS or chiral primaries of the gauge theory i.e they do not have anomalous dimensions. The dilatation operator in the half BPS sector of chiral primaries is then the tree level dilatation operator, or the matrix Harmonic oscillator

$$H_{cp} = \mathrm{tr}(A^{1\dagger} A_1). \tag{1.3}$$

It is understood that one has mapped the chiral primaries to the states of the dilatation operator with $A^{1\dagger}$ being the creation operator for the Z_1 type of excitations. The matrix harmonic oscillator simply counts the number of fields sitting inside the state. As suggested in [3] it is more sensible to think of the above Hamiltonian as a gauged matrix model:

$$H_{cp} = \frac{1}{2}\mathrm{tr}\left((D_t X_1)^2 + X_1^2\right) : D_t X = \dot{X} + [G, X] \tag{1.4}$$

where G is a gauge connection. The gauge invariance is simply inherited from original gauge theory for which the dilatation operator is the radial Hamiltonian. Fixing the gauge $G = 0$ is a projection of the dynamics on to the singlet states, and the Hamiltonian simply reduces to that of a matrix oscillator. Following standard techniques the gauge choice reduces the number of degrees of freedom of the gauge theory from N^2 to N, and the gauge invariant microscopic dynamics of the half BPS sector can be reformulated as the dynamics of the N eigenvalues of the matrix X_1. The change of variables form the matrix elements to the eigenvalues introduces a Jacobian, which can be absorbed in a redefinition of the wave function which subsequently becomes antisymmetric enabling us to interpret the eigenvalues x_i as fermions. The hamiltonian for the eigenvalues is simply the free one

$$H_{CP} = \frac{1}{2}\sum_i \left(-\frac{\partial^2}{\partial x_i^2} + x_i^2\right) \tag{1.5}$$

One thus has an interpretation of the gauge invariant degrees of freedom of the scalar half BPS sector of $\mathcal{N}$=4SYM in terms of N free fermions described by (1.5). The free fermion picture has proved to be extremely useful in understanding non-perturbative aspects of the AdS/CFT correspondence. For instance a precise map between half BPS geometries and the phase space density of free fermions has been proposed by [4] while several large excitations of the free Fermions have been related to BPS branes and giant gravitons in the dual string theory [13, 6, 3].

The existence of two equivalent descriptions of the matrix harmonic oscillator has also been viewed as an example of an exact realization of open/closed duality in $\mathcal{N}$ =4 SYM [3]. The description of the states of the matrix model in terms of products of traces of the matrix creation operators has been viewed as a closed string description of the dual string theory. An operator such as

$$(\mathcal{B}_n)^\dagger = \mathrm{tr}((A^\dagger)^n) \tag{1.6}$$

can be viewed as a creation operator for a closed string mode of energy n. A typical matrix model state of energy n can then be described by all the partitions of the number n into $n_1 \cdots n_i$ such that $n_1 \geq n_2 \cdots \geq n_i$. To each partition one may associate a Young

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Tableaux having columns with $n_1 \cdots n_i$ boxes. This has been regarded as a realization of the description of the degeneracy of the dual closed string excitations on [6, 3]. As a matter of fact a world sheet/string sigma model description of the matrix harmonic oscillator has also been found recently [12], though the connection of the world sheet description of the matrix oscillator and string theory on $AdS_5 \times S^5$ probably requires further study.

On the other hand, the description of the states of the matrix model in terms of eigenvalues: the free Fermion picture has been regarded as a an open string /D brane description of the half BPS sector of the AdS/CFT correspondence. The classic Bosonization result that relates the degeneracies of states of a free fermion system in $1+1$ dimensions to that of a chiral Boson, where once again the degeneracies are counted by the number of partitions of the integer energy levels has been regarded as an open string description of the BPS spectrum of the dual string theory.

Of special interest in recent investigations has been the description of large excitations i.e excitations with energies of $O(N)$. These large departures from the Fermi sea, known as the giant gravitons correspond to operators built out of determinants and sub-determinants rather than traces [6, 3]. For instance the gauge theory description of the maximal BPS giant corresponds to the state

$$\epsilon_{i_i \cdots i_N} \epsilon^{j_1 \cdots j_N} A^{1\dagger i_1}_{j_1} \cdots A^{1\dagger i_N}_{j_N} |0\rangle \tag{1.7}$$

Such large non-perturbative BPS excitations provide one with a gauge theory description of BPS branes on the dual string geometry. The free Fermion picture gives a particularly simple and elegant description of the BPS giants: they simply correspond to exciting an eigenvalue from the bottom of the Fermi sea to the top. The Fermionic description sheds light on a host of issues related to the AdS/CFT correspondence. For instance the vibration frequencies of (BPS) giant gravitons computed by a world-volume computation [39, 40]can be reproduced in the gauge theory language using by solving the matrix harmonic oscillator. The matrix oscillator description can also be used to clarify a host of issues regarding the gauge theory duals of non-perturbative string states. Several such recent interesting developments have been discussed in [14 − 21].

The class of BPS operators described in the brief review above are all charged under a U(1) of the SO(6) R symmetry group of the gauge theory. However, it is just as natural to consider operators that carry several other charges. These would correspond to protected operators that involve more than a single scalar field inside a trace. Hence, a natural question that arises out of this line of investigation is how the free fermion picture changes in a systematic way once BPS excitations involving multiple fields are considered. For instance, in the particular case of the $su(2|3)$ sector considered here, one could have operators such as

$$\mathrm{tr}(Z_1^n \Phi) \tag{1.8}$$

where Φ can be any one of Z_2, Z_3, Ψ_1, Ψ_2. These operators, being the supersymmetry descendants of $\mathrm{tr}Z_1^n$ are also protected. Maximal giant gravitons such as

$$\epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} (A^{\dagger 1})^{i_1}_{j_1} \cdots (A^{\dagger 1})^{i_{N-1}}_{j_{N-1}} (A^{\dagger \alpha})^{i_N}_{j_N} |0\rangle \tag{1.9}$$

where $A^{\dagger\alpha}$ corresponds to an impurity excitation also fall into the same category of protected operators. Similarly, one could build protected operators with multiple 'impurity ' fields inside a single trace. A particularly simple example would be

$$\mathrm{tr}(Z_1\Phi\Phi). \tag{1.10}$$

The principal question that we shall address in this paper is what the appropriate generalization of the free Fermion picture is when generic protected operators that involve an arbitrary number of fields are considered. This question is also relevant from the point of view of understanding the role of supersymmetry in the description of BPS dynamics as many-body theories. Since the $su(2|3)$ sector contains some amount of supersymmetry one can hope to learn how the supersymmetry manifests itself in the open-string picture. The role of supersymmetry is not obvious at the level of the free Fermion system[1].

The dilatation operator, restricted to the set of BPS operators in the $su(2|3)$ sector is nothing but the sum of five decoupled harmonic oscillator Hamiltonians.

$$H' = \sum_{i=1}^{3} \mathrm{tr}\left(A^{i\dagger}A_i\right) + \frac{3}{2}\sum_{I=1}^{2} \mathrm{tr}\left(\Psi^{I\dagger}\Psi_I\right). \tag{1.11}$$

The factor of $\frac{3}{2}$ in front of the Fermionic Hamiltonian is nothing but the engineering dimension of the Fermionic fields of the gauge theory. In what is to follow, we shall subtract a term proportional to the Fermion number operator and work with

$$H = H' - \frac{1}{2}\sum_{I=1}^{2} \mathrm{tr}\left(\Psi^{I\dagger}\Psi_I\right). \tag{1.12}$$

Since the dilatation operator does not change the Fermion number, H and H' carry the same information, and in various analyses that are carried out in this paper, we shall give explicit prescriptions for understanding various features (such as degeneracies, Yangian symmetries etc) of H' from the studies of the corresponding properties of H. To simplify the notation we shall write the Hamiltonian as

$$H = \sum_{\alpha=1}^{5} \mathrm{tr}\left(A^{\alpha\dagger}A_\alpha\right). \tag{1.13}$$

It will be understood that $\alpha = 1, 2, 3$ correspond to Bosonic matrices while $\alpha = 4, 5$ correspond to Fermionic ones.

Generalization of the closed string point of view from the half BPS sector involving a single matrix oscillator to the $su(2|3)$ sector follows immediately. The closed string excitations are identified as states of the matrix model formed out of products of traces of the creation operator acting on the vacuum, i.e., they are states of the form

$$\prod_n \mathrm{tr}\left(A^{\dagger\alpha_1}\cdots A^{\dagger\alpha_n}\right)|0\rangle \tag{1.14}$$

[1]For a recent parallel line of investigation into the study of multi-charge giants, see [24]

Of course when all the $\alpha_i = 1$, we revert back to the half BPS sector of chiral primaries. In this point of view one can work out the degeneracies corresponding to a states with given energies much along the same lines as the analysis involving a single field Z_1. The analysis is a little more involved but it can be carried out nevertheless using the Polya formulae for counting the number of distinct 'words' formed out of a certain number of 'letters' in an 'alphabet': in our case five[2]. However the open string description i.e the analog of the eigenvalue dynamics of the multi-matrix model is not obvious at all. An open string description of the full $su(2|3)$ sector would require an understanding of how the free Fermion picture of the chiral primary states changes in a systematic way once multi-charge BPS excitations are allowed. Such BPS excitations would correspond to the impurity fields $\alpha = 2 \cdots 5$ to be present inside a single trace, the simplest of which would correspond to states such as

$$\mathrm{tr}\left((A^{1\dagger})^n A^{\alpha\dagger}\right)|0\rangle \tag{1.15}$$

where α is one of the impurity fields $2 \cdots 5$. As mentioned before, one can in general have BPS states with a large number of impurity fields inside a single trace. Clearly, the problem of deriving an open string description of the multi-charge BPS states amounts to finding a description of the dynamics of the eigenvalues of the matrix X_1 in the background of the impurity fields.

In the present work we shall take a step in the direction of understanding this problem. We shall be able to show that in the presence of impurity excitations the eigenvalues can be understood as Fermions with internal/spin degrees of freedom. They will turn out to interact with each other through spin dependent inverse square interactions. As a matter of fact we shall be able to formulate the dynamics of the eigenvalues in the background of a arbitrary number of impurity excitations in terms of generalizations of the celebrated Calogero systems. Furthermore, we shall also be able to show that for case of the simplest departure from the Free Fermion picture involving the study of BPS operators consisting of a single impurity field inside a trace the dynamics reduces to the well known super symmetric rational Calogero model. We shall study this sector in some detail, as it has all the features of the most general particle mechanics that one can encounter in the study of the multi-charge BPS operators. The relation between super-Calogero models and matrix models was made by Dabholkar [26], where the super Calogero model was shown to be a consistent truncation of the Marinari-Parisi model. In the case at hand we shall be able to see that a similar truncation has a natural interpretation in the study of $\mathcal{N} = 4$ SYM as the restriction of the dynamics of the dilatation operator to protected operators of a particular type. After setting up the correspondence between multi charge operators and the super Calogero system we shall recover the complete spectrum and degeneracies of the BPS excitations of the matrix model within the framework of the Calogero system. This might be regarded as the realization of an open/closed duality for multi-charge protected operators much along the same lines as the one between the single matrix oscillator and the free Fermion system.

[2] We shall refer the reader to [41] for a simple discussion of Polya counting applied to matrix harmonic oscillators

Not all the excitations of the matrix model correspond to protected operators of the gauge theory. However, the non-BPS excitations of the matrix model are bona-fide local composite operators of the gauge theory. Other than the sector of protected operators, the matrix oscillators also provide one with a non-perturbative definition of the gauge theory dilatation operator in the limit of $g_{\mathrm{YM}}^2 \to 0$. As is well known, the string dual to the free gauge theory is notoriously hard to pin down. Thus the ulterior motive behind our study of the tree level dilatation operator is that perhaps its gauge fixed form can be utilized to discover the string theory which is relevant in the limit of zero Yang-Mills coupling. Though we do not make an attempt at finding the string theory, we do identify and study operators that have the curious property of being protected in the large N limit, while at finite values of N they turn out to be BMN like operators with small anomalous dimensions. The parameter that governs their BPS condition is $\frac{1}{N}$. It is in the study of these operators that we find that the dilatation operator takes on the familiar form of the Calogero model.

Apart from analyzing the spectrum and the open/closed duality, we shall also use the Calogero system to investigate the hidden symmetries that lead to its integrability. The motivation for doing this is the use of the protected sector of the gauge theory as a probe to understand whether or not any of the integrable structures (such as Yangian symmetries) that are present in the string sigma model survive the supergravity limit. Interestingly enough, for the case of the Calogero model we shall be able to see that the underlying symmetry is not an Yangian but rather its loop algebra. In the light of the fact that the loop algebra can be regarded as a classical limit of the Yangian algebra (the symmetry of the string sigma model) it is reasonable to expect it to be the symmetry of the classical limit of the string theory. In the simplest non-trivial example that we study in this paper, this expectation is indeed realized.

After a detailed description of operators involving a single impurity field inside a matrix trace in terms of the Calogero model, we shall describe the dynamics of the most general (multi-charge) protected operators. The particle mechanics in the general case will turn out to be governed by a particlar (integrable) generalization of the rational Calogero systems known as the Euler-Calogero systems [27]. We shall be able to exploit the integrability of these systems to understand the spectrum and degeneracies of the most general multi-charge BPS operators as well.

We shall finally conclude with comments on some unresolved issues and directions for future explorations.

2. Multi-matrix harmonic oscillators

In this section we shall present an overview of the techniques that are necessary to have a gauge fixed description of a collection of matrix harmonic oscillators. The starting point is a system of d Hermitian matrices, $(X^\alpha)^i_j$, $\alpha = 1 \cdots d$. $i, j = 1 \cdots N$. Keeping in mind the $su(2|3)$ sector, we shall let $d = 5$, with $d = 1 \cdots 3$ being bosonic and the rest fermionic. The Hamiltonian for the matrix model will be taken to be a sum of harmonic oscillators,

$$H = \sum_\alpha \mathrm{tr} \frac{1}{2} \left(\Pi^\alpha \Pi^\alpha + X^\alpha X^\alpha \right) \tag{2.1}$$

Π is the momentum conjugate to X, and the canonical commutation relations are,

$$[(X^\alpha)^i_j, (\Pi^\beta)^k_l]_\pm = i\hbar\delta^{\alpha,\beta}\delta^k_j\delta^i_l \tag{2.2}$$

One could go to the Holomorphic basis of creation and annihilation operators, in which the Hamiltonian becomes

$$H = \sum_\alpha tr\left(A^{\dagger\alpha}A_\alpha\right) \tag{2.3}$$

2.1 Generalized Calogero systems

We want to write the system of Harmonic oscillators in a basis in which one of the matrices, X^1 is diagonal. Changing variables from the matrix elements to the eigenvalues in matrix models involving several matrices is in general hard to accomplish. However, when only one matrix is diagonalized this becomes tractable. The matrices do not couple to each other, so the dynamics of the eigenvalues of a single matrix is that of a spin-Calogero type, where the role of spin is played by the generators of unitary conjugations [29]. This creates an effective coupling to the remaining matrices, as the Gauss law relates these generators to those of the remaining matrices. Such a reduction was worked out by Ferretti in [27] in the context of the Marinari-Parisi model(see also [28]). Below we outline the procedure for our case.

Let us denote the diagonal elements of X^1 by x_i

$$X^1 = U^\dagger x U. \tag{2.4}$$

Furthermore, let us denote the oscillators in this basis by lower case letters

$$(a^\alpha)^i_j = (UA^\alpha U^\dagger)^i_j, \qquad (a^{\dagger\alpha})^i_j = (UA^{\dagger\alpha}U^\dagger)^i_j, \qquad \alpha \neq 1 \tag{2.5}$$

Let us now proceed to write down the Hamiltonian of the decoupled set of oscillators as a generalized Calogero system. We are going to treat all the oscillators other than the first one as impurities so

$$H = H_1 + H_{\text{Imp}} \tag{2.6}$$

where H_1 denotes the Hamiltonian for the first oscillator. H_{Imp} can be written easily enough as

$$H_{\text{Imp}} = \sum_{\alpha\neq 1} \text{tr}(a^{\dagger\alpha}a_\alpha). \tag{2.7}$$

To write the first oscillator in the eigenvalue basis one starts with the metric on the space of Hermitian matrices

$$ds^2 = \sum_i dx_i dx_i + \sum_{i\neq j}(x_i - x_j)^2 \omega^{\star p}_q \omega^q_p. \tag{2.8}$$

The one forms ω are defined as

$$\omega^i_j = (dU)^i_k (U^\dagger)^k_j. \tag{2.9}$$

Similarly, one also has the dual vector fields $\mathcal{L}$

$$\mathcal{L}^i_j = U^i_m \frac{\partial}{\partial U^m_j} \tag{2.10}$$

that obey the U(N) Lie algebra

$$[\mathcal{L}^i_j, \mathcal{L}^k_l] = \delta^k_j \mathcal{L}^i_l - \delta^i_l \mathcal{L}^k_j \tag{2.11}$$

Using the metric the momentum operator can be written as

$$\frac{\partial}{\partial X^j_i} = (U^\dagger)^i_k \pi^k_l U^k_j \tag{2.12}$$

where

$$\pi^i_j = \frac{\partial}{\partial x_i}\delta^i_j + \frac{1-\delta^i_j}{x_i - x_j}\mathcal{L}^i_j \tag{2.13}$$

We can now write

$$H_1 = \sum_i \frac{1}{2}\left(-\frac{\partial}{\partial x_i^2} + x_i^2\right) + \frac{1}{2}\sum_{i\neq j}\left(\frac{\mathcal{L}^i_j \mathcal{L}^j_i}{(x_i - x_j)^2}\right) \tag{2.14}$$

This is clearly a generalized U(N) spin-Calogero system. However, we want to formulate the particle mechanics completely in terms of the microscopic degrees of freedom which the are N eigenvalues x_i and the remaining matrix oscillators $a^{\alpha i}_j, a^{\dagger\alpha i}_j, \alpha \neq 1$. To do that we note that U(N) singlet states of the particle mechanical system would generically be of the kind

$$\Psi^{i_1\cdots i_n}_{j_i\cdots j_n}(x)\Pi^n_{k=1}(a^{\dagger\alpha_k})^{j_k}_{i_k}|0\rangle. \tag{2.15}$$

Ψ is an U(N) tensor which depends on the N eigenvalues x_i. The dependence of the state on the $\frac{N(N-1)}{2}$ angular degrees of freedom is contained in $(a^{\dagger\alpha})^i_j$ which depend on the angular coordinates through (2.5). It may now be easily verified that

$$[\mathcal{L}^i_j, (a^{\dagger\alpha})^a_b] = \left[\sum_\beta \left((a^{\dagger\beta})^i_l (a^\beta)^l_j - (a^{\dagger\beta})^l_j (a^\beta)^i_l\right), (a^{\dagger\alpha})^a_b\right]. \tag{2.16}$$

This identity follows from noticing that

$$[\mathcal{L}^i_j, (a^{\dagger\alpha})^a_b] = \delta^a_j (a^{\dagger\alpha})^i_b - \delta^i_b (a^{\dagger\alpha})^a_j \tag{2.17}$$

which may be compared with the explicit action of the angular derivatives on the angular coordinates present in the definition of $(a^{\dagger\alpha})^a_b$. We may thus replace the vector fields appearing in the Hamiltonian by the matrix operators, i.e.

$$\mathcal{L}^i_j = \sum_\beta \left((a^{\dagger\beta})^i_l (a^\beta)^l_j - (a^{\dagger\beta})^l_j (a^\beta)^i_l\right) \tag{2.18}$$

From now on it will always be implied (unless stated explicitly) that the vector fields have been replaced by their oscillator realization (2.18). We have thus completed writing the

Hamiltonian in terms of the degrees of freedom available to us in the basis in which the first matrix is diagonal. The Hamiltonian being

$$H = \sum_i \frac{1}{2}\left(-\frac{\partial}{\partial x_i^2} + x_i^2\right) + \frac{1}{2}\sum_{i\neq j}\left(\frac{\mathcal{L}^i_j \mathcal{L}^j_i}{(x_i - x_j)^2}\right) + \sum_{\alpha\neq 1} \mathrm{tr}(a^{\dagger\alpha} a_\alpha) \tag{2.19}$$

with

$$\mathcal{L}^i_j = \sum_{\beta\neq 1}\left((a^{\dagger\beta})^i_l (a^\beta)^l_j - (a^{\dagger\beta})^l_j (a^\beta)^i_l\right) \tag{2.20}$$

2.2 Residual constraints on physical states

A typical state $|\psi\rangle$ of the Calogero system is

$$|\psi\rangle = \psi^{i_i\cdots i_m}_{j_i\cdots j_m}(x)(a^{\dagger\alpha_1})^{j_1}_{i_1}\cdots(a^{\dagger\alpha_1})^{j_m}_{i_m}|0\rangle, \tag{2.21}$$

where ψ is a $\mathrm{U}(N)$ tensor. Not all the states of the many-body theory are allowed states of the gauge fixed matrix model. The states have to be invariant under the residual gauge symmetry left over even after carrying out the $\mathrm{U}(N)$ rotation to the space of eigenvalues of X^1. One must ensure that the diagonal subgroup of $\mathrm{U}(N) = \mathrm{U}(1)^N$ that leaves the eigenvalues invariant also leave the state invariant. So physical states have to satisfy the constraint

$$\mathcal{L}^i_i|\psi\rangle = \sum_\beta\left((a^{\dagger\beta})^i_l (a^\beta)^l_i - (a^{\dagger\beta})^l_i (a^\beta)^i_l\right)|\psi\rangle = 0 \tag{2.22}$$

The model described above can be regarded as a generalization of the well known spin-Calogero models. Unlike the usual Calogero models the model above has a very large number of 'spin' degrees of freedom. The model is still Fermionic, as the overall wave function is antisymmetric under the exchange of the particles. Thus the Free fermion picture of BPS operators carrying a single $\mathrm{U}(1)$ charge seems to be a replaced by a picture of interacting Fermions. The Fermions carry an internal spin degree of freedom and interact through spin dependent inverse square interactions.

The classical limits of such generalized $\mathrm{SU}(N)$ Calogero systems have been studied in the literature in the past for independent reasons and they are referred to as Euler-Calogero systems. We shall adhere to this terminology in the present work as well. These systems are also known to be integrable at the classical level [34, 35]. Later in paper, we shall be able to utilize the connection to matrix oscillators to confirm the quantum integrability of these models and understand their spectrum.

The $\mathrm{SU}(N)$ Calogero model is known to contain various Calogero models with fewer number of spin degrees of freedom as consistent truncation of its dynamics to suitable chosen subspaces of its full Hilbert space. For a discussion of such reductions in the context of trigonometric Calogero models we shall refer to [29, 30, 33]. Thus it is of interest to study whether or not the usual Calogero models play any special role in the understanding of BPS operators of the gauge theory. In the following section we shall show that this is indeed true.

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3. A Dabholkar-like truncation

The first class of operators that we shall look at are the ones that have at the most only a single impurity excitation located inside a single trace. Moreover, we shall restrict ourselves to the case where the impurities are Fermionic. These are states of the form

$$\frac{1}{\sqrt{N^m}}\mathrm{tr}\left((A^{\dagger 1})^m\right)\frac{1}{\sqrt{N^{n_1+1}}}\mathrm{tr}\left((A^{\dagger 1})^{n_1}\Psi^{\dagger\alpha_1}\right)\cdots\frac{1}{\sqrt{N^{n_i+1}}}\mathrm{tr}\left((A^{\dagger 1})^{n_i}\Psi^{\dagger\alpha_i}\right)|0\rangle. \quad (3.1)$$

$\alpha_1\cdots\alpha_i = 1,2$. An interesting aspect of these states is that they are protected in the large N limit.

These states when written in the basis in which X^1 is diagonal would generically appear as

$$\prod_m \Psi(x_1\cdots x_N)^{i_1\cdots i_m}(\mathcal{A}^{\dagger\alpha_1})_{i_1}\cdots(\mathcal{A}^{\dagger\alpha_m})_{i_m}|0\rangle + O\left(\frac{1}{N}\right) \quad (3.2)$$

where,

$$(\mathcal{A}^{\dagger\alpha})_i = (\psi^{\dagger\alpha})^i_i. \quad (3.3)$$

are the excitations corresponding to the diagonal matrix elements of the impurity creation operators in the rotated basis. It is possible to perform a consistent truncation of the particle mechanical system to a Hilbert space $\mathcal{H}_d$ spanned by states of the above type.

To see that this truncation is consistent one needs to show that $\mathcal{H}_d$ is closed under the action of the Hamiltonian. This is obviously the case as the Hamiltonian does not change the number of impurity fields inside the traces.

As the $su(2|3)$ sector has two Fermionic degrees of freedom, one can consider states for which the impurities correspond to only one of the two Fermionic degrees of freedom available to us, that is, either Ψ^1 or Ψ^2, which we will simply call Ψ. This is the so-called $su(1|1)$ sector of the gauge theory, and operators formed out of the two degrees of freedom, X^1 and Ψ, are also closed under dilatation. Furthermore, the quartic spin interaction term of the Euler-Calogero model assumes a much simpler and familiar form within this truncated subspace, as it can be represented by a graded exchange operator

$$\mathcal{L}^i_j\mathcal{L}^j_i = \frac{1}{2}(1-\Pi_{i,j}). \quad (3.4)$$

$\Pi_{i,j}$ is a graded permutation operator that exchanges the spins at the lattice sites i and j while picking up a negative sign if both the spins happen to be Fermionic.

To see how this arises, assume that the angular SU(N) conjugation generators $\mathcal{L}$ are in a representation generated by

$$\mathcal{L}^i_j = b^\dagger_i b_j - f^\dagger_j f_i \quad (3.5)$$

where $b_i, b^\dagger_i$ and $f_i, f^\dagger_i$ are a set of bosonic and fermionic oscillator ladder operators, respectively. The above construction embeds in the oscillators' Fock space all totally symmetric products of the fundamental times all totally antisymmetric products of the antifundamental of SU(N). The residual physical constraint reads

$$\mathcal{L}^i_i|\psi\rangle = (b^\dagger_i b_i - f^\dagger_i f_i)|\psi\rangle = 0 \quad (3.6)$$

which implies that the Boson and Fermion number for each index i are both equal to 0 or 1. This realizes the group SU(1|1) on each site i, acting upon the 'spin' states of the site labelled by their Fermion number $0, 1$. Using the above condition, it is an easy matter to show that $\mathcal{L}^i_j \mathcal{L}^j_i$ reduces to the graded exchange operator $1 - \Pi_{i,j}$ when it acts on physical states.

To complete the demonstration, we remark that the representation of $\mathcal{L}^i_j$ carried by the states (3.1) is exactly the one embedded in the above construction. Indeed, writing $A^{\dagger 1} = A^\dagger$, gauge invariant states in terms of $b^\dagger_i$ and $f^\dagger_i$ are generated through the action of operators

$$b^\dagger_i (A^\dagger)^n_{ij} f^\dagger_j = \mathrm{tr}\left((A^\dagger)^n f^\dagger b^\dagger\right) \tag{3.7}$$

where we view $b^\dagger_i$ as a row vector and $f^\dagger_i$ as a column vector. This is identical to the operators appearing in (3.1) upon identifying $\Psi^\dagger$ with $f^\dagger b^\dagger$ (both operators are fermionic and have the same SU(N) transformation properties). In this realization, however, there are no multiple impurities per trace, since

$$\mathrm{tr}\left((A^\dagger)^n f^\dagger b^\dagger (A^\dagger)^m f^\dagger b^\dagger\right) = \mathrm{tr}\left((A^\dagger)^n f^\dagger b^\dagger\right) \mathrm{tr}\left((A^\dagger)^m f^\dagger b^\dagger\right) \tag{3.8}$$

So the space spanned by single impurity traces is isomorphic to the above SU(1|1) spin representation. Further, in the X^1 diagonal (eigenvalue) representation, physical states arise through the action of $b^\dagger_i f^\dagger_i$ for each eigenvalue. We can thus identify $(\mathcal{A}^\dagger)_i = (\Psi^\dagger)^i_i$ with the above operator, obtaining a correspondence with Dhabolkar-like states (3.2).

By using the formalism developed above, the Hamiltonian in the SU(1|1) sector can be written as

$$H = \sum_i \frac{1}{2}\left(-\frac{\partial}{\partial x_i^2} + x_i^2\right) + \frac{1}{2}\sum_{i \neq j}\left(\frac{1 - \Pi_{i,j}}{(x_i - x_j)^2}\right) + \sum_j \mathcal{A}^{\dagger j}\mathcal{A}_j \tag{3.9}$$

By using the fermionic form of the graded permutation operator,

$$\frac{1}{2}\left(\mathcal{A}^{\dagger i}\mathcal{A}_i + \mathcal{A}^{\dagger j}\mathcal{A}_j - \mathcal{A}^{\dagger i}\mathcal{A}_j - \mathcal{A}^{\dagger j}\mathcal{A}_i\right) = 1 - \Pi_{i,j} \tag{3.10}$$

we can recast the above Hamiltonian as

$$H = \sum_i \left(-\frac{1}{2}\frac{\partial}{\partial x_i^2} + \mathcal{A}^{\dagger i}\mathcal{A}_i + \frac{1}{2}x_i^2\right) + \frac{1}{2}\sum_{i \neq j}\left(\frac{\mathcal{A}^{\dagger i}\mathcal{A}_i - \mathcal{A}^{\dagger i}\mathcal{A}_j}{(x_i - x_j)^2}\right) \tag{3.11}$$

This is nothing but the supersymmetric rational Calogero model. This very model has appeared in the analysis of superstrings in two dimensions where it was shown by Dabholkar to be a consistent truncation of the Marinari-Parisi model [26]. The truncation that we perform is similar to the one carried out by Dabholkar, however, it is to be kept in mind that the eigenstates of the Calogero system correspond to protected operators of the gauge theory only in the large N limit. Another gratifying aspect of the present analysis, which will be made clear in the following sub-section is that one can have a one to one map between the excitations of the Calogero system and the those of the matrix model. Such

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a map between the open and closed string pictures is slightly obscure in the approach pioneered in [26].

We thus see that the super-Calogero model is relevant to the study of $\mathcal{N}$=4 SYM as being the natural generalization of the theory of free Fermions which is relevant for the study of BPS operators with no impurities. The Calogero model is still a theory of Fermions as the overall wave function is antisymmetric under the exchange of the particles, but the Fermions are no longer free and they carry an internal spin degree of freedom.

3.1 From the matrix model to the Calogero system

We shall now elaborate on the connection of the Calogero model to the matrix model, and in the process provide an alternative explanation for why it was reasonable to replace the quartic spin interaction term by the quadratic graded permutation operator. The simplest way to understand the connection to the super Calogero model is by looking at the spectrum of the matrix model in the subspace considered above. It is quite clear that the spectrum of the matrix model is the same as that of a system of Bosonic and Fermionic oscillators with frequencies given by integers. One can introduce Bosonic an Fermionic creation operators B_n and F_k which create oscillator states of energies n and k respectively. It is then possible to map the matrix model states to oscillator states using the following identification:

$$B_n|0\rangle \leftrightarrow \frac{1}{\sqrt{N^n}} tr(A^\dagger)^n|0\rangle \tag{3.12}$$

for Bosonic states and

$$F_k|0\rangle \leftrightarrow \frac{1}{\sqrt{N^k}} \left[tr(A^\dagger)^{k-1}\Psi^\dagger\right]|0\rangle \tag{3.13}$$

for the Fermionic ones. The Bosonic and Fermionic oscillators can be taken to be related to each other through a supersymmetry algebra given by:

$$\begin{aligned}
&[F_m, F_n]_+ = 0, [B_m, F_n] = 0, [B_m, B_n] = 0\\
&[Q, F_m]_+ = 0, [Q^\dagger, F_n]_+ = B_n, [H, F_n] = nF_n\\
&[Q, B_n] = 2nF_n, [Q^\dagger, B_n] = 0, [H, B_n] = nB_n
\end{aligned} \tag{3.14}$$

H in the above set of equations is the Hamiltonian for the free super oscillators whose frequencies are given by integers. But this is nothing but the rational super-Calogero model in disguise. The super-Calogero model and its spectrum has been studied in various papers in the past, see for example [36, 37], and it is known that it can be brought to a form where the Hamiltonian becomes a collection of free super oscillators by a similarity transform. We shall now summarize the similarity transformation that brings the Calogero model to the form of the super-oscillators for the sake of completeness.

The Calogero model has a manifest supersymmetry which is generated by

$$\begin{aligned}
\mathcal{Q} &= \sum_i \mathcal{A}^{\dagger i}\Pi_i\\
\mathcal{Q}^\dagger &= \sum_i \mathcal{A}_i\Pi_i^\dagger
\end{aligned} \tag{3.15}$$

where Π_i are the coupled momentum operators [25]

$$\Pi_i = p_i - iW_i, \Pi_i^\dagger = p_i + iW_i. \tag{3.16}$$

$p_i = -i\frac{\partial}{\partial x_i}$ while $W_i = \frac{\partial W}{\partial x_i}$ with W being the superpotential

$$W = -\ln \Pi_{i<j}(x_i - x_j) + \frac{1}{2}\sum_i x_i^2 \tag{3.17}$$

Some straightforward algebra shows that (up to a constant term) the Hamiltonian can be written in a manifestly supersymmetric form

$$H = \frac{1}{2}[\mathcal{Q}, \mathcal{Q}^\dagger]_+ \tag{3.18}$$

The ground state has Fermion number = 0, and it is the same as that of the free Fermion system:

$$\Omega = \prod_{i<j}(x_i - x_j)e^{-\frac{1}{2}\sum_i x_i^2}|0\rangle \tag{3.19}$$

The higher excitations above this ground state can be understood in a purely algebraic fashion by mapping the Calogero system to a system of free super-oscillators with frequencies given by integers $1 \cdots N$. The explicit form of the similarity transformation that maps the super-Calogero system to the system of free super-oscillators has been worked out in detail in [36], and we shall gather together the relevant results that are necessary for understanding the degeneracies. One can introduce the Bosonic and Fermionic raising operatorsB_n and F_n

$$\frac{1}{2^n}B_n = \sum_i \Gamma^{-1}x_i^n\Gamma, \frac{1}{2^{n-1}}F_n = \sum_i \Gamma^{-1}\mathcal{A}_i^\dagger x_i^{n-1}\Gamma, \tag{3.20}$$

where

$$\Gamma = e^{\frac{S}{2}}(-\ln\Omega) \tag{3.21}$$

and

$$S = \frac{1}{2}\sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}\frac{1}{x_i - x_j}\frac{\partial}{\partial x_i} - \sum_{i\neq j}\frac{1}{(x_i - x_j)^2}(\mathcal{A}_i^\dagger\mathcal{A}_i - \mathcal{A}_i^\dagger\mathcal{A}_j) \tag{3.22}$$

Similarly, one could also apply the similarity transformation to the supercharges,

$$Q = \Gamma^{-1}\mathcal{Q}\Gamma, Q^\dagger = \Gamma^{-1}\mathcal{Q}^\dagger\Gamma \tag{3.23}$$

Some straightforward but lengthy algebraic computations yield that the algebra obeyed by the raising operators and the supercharges is (3.14). Thus, we see that the truncation of the matrix model to states involving only one impurity inside a single trace can be described by the super-Calogero model.

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3.2 Degeneracies

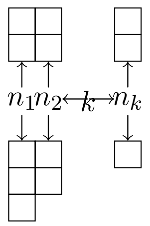
This algebraic structure makes the spectrum and the associated degeneracies of the model extremely transparent. As in the free Fermion picture the degeneracies of the Bosonic states are counted by partitions of integers. The states

$$B_n|0\rangle \text{and} \prod_{i=l}^{l} B_{n_i}|0\rangle, \sum_i n_i = n \tag{3.24}$$

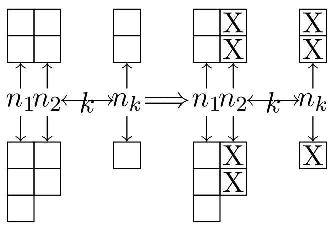
are degenerate which is the open string description of the degeneracies between matrix model states

$$\mathrm{tr}[(A^\dagger)^n]|0\rangle \text{and} \prod_i \left[\mathrm{tr}(A^\dagger)^{n_i}\right]|0\rangle, \sum_i n_i = n. \tag{3.25}$$

Making a choice of ordering such that $n_1 \geq n_2 \geq n_3 \cdots$ one has the result that the states with energy n can be represented by Young diagrams with n boxes. For instance the state with energy n corresponds to a Young diagram with columns of length $n_1 \geq n_2 \geq n_3 \cdots$.



For the full Calogero system, there are further degeneracies due to the fact that every B_n excitation is degenerate to a F_n excitation; which is simply a consequence of the manifest supersymmetry. Thus a state with energy n can, once again be represented by a Young diagram with n boxes, but each one of the columns (of length n_i) now has the option of corresponding to either a B_{n_i} or a F_{n_i} excitation. We can denote the columns corresponding to the F excitations by drawing them with boxes with crosses as depicted below. Hence the complete set of degenerate states for the Calogero model, corresponding to an excitation of energy n, are described by first forming all the Young diagrams corresponding to the partitions of n. The action of the supersymmetry generators can then be described by replacing the columns with the ones containing crossed boxes, one column at a time. For example, the effect of replacing two columns with crossed ones is depicted below.



One also has to impose the rule that in any given Young diagram one can have at the most one 'crossed' column of a given length. This simply follows from the fact

$$F_{n_i}F_{n_i} = 0 \tag{3.26}$$

To each such partition, one can associate a state of the Calogero model. The naive association of states to partitions would be to associate the the appropriate an oscillator excitation to every column of the Young diagram, i.e., an excitation of the $B_n(F_n)$ type for each uncrossed (crossed) column of length n. Although the states so formed would be bona fide eigenstates of the Calogero Hamiltonian, they will not diagonalize the Higher conserved charges of the Calogero system. This is the analog of the difference between the string basis and the basis formed my taking the Slater determinants of the various Hermite polynomials for the free Fermion system [8]. However, the eigenstates that diagonialize all the mutually commuting charges of the Calogero system were identified in [38] and their relation to the partitions described above was also made clear in the same paper. Since, we shall not be involved in the diagonalization of the higher charges in the present work, we shall refer to [38] for further details of the construction of eigenfunctions.

Having enumerated the degeneracies of the Calogero model we can now proceed to apply these results to the dilatation operator H'. The dilatation operator differs from the Calogero system by a term proportional to the Fermion number operator. However the above discussion can be easily generalized to understand its degeneracies as well. In the matrix model language, the Hamiltonian is

$$H' = \mathrm{tr}\left(A^\dagger A + \frac{3}{2}\Psi^\dagger\Psi\right), \tag{3.27}$$

and the factor of three halves in front of the Fermion number operator is due to the fact that the dilatation operator measures the conformal dimensions of the gauge theory composite operators and the Fermions have a bare conformal dimension of $\frac{3}{2}$ while that for the scalars os 1. In the basis, where the position space matrix corresponding to A is diagonal, the dilatation operator $H + \frac{1}{2}\mathcal{A}^{\dagger i}\mathcal{A}_i$ is:

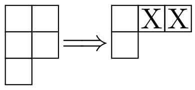
$$H' = \sum_i \left(-\frac{1}{2}\frac{\partial}{\partial x_i^2} + \frac{3}{2}\mathcal{A}^{\dagger i}\mathcal{A}_i + \frac{1}{2}x_i^2\right) + \frac{1}{2}\sum_{i\neq j}\left(\frac{\mathcal{A}^{\dagger i}\mathcal{A}_i - \mathcal{A}^{\dagger i}\mathcal{A}_j}{(x_i - x_j)^2}\right). \tag{3.28}$$

The previous discussion about states being labeled by F and B type oscillators goes through but the degeneracies are to be counted in a somewhat different manner. From the Hamiltonian, it is clear that three Bosonic excitations have the same energy as two Fermionic ones, thus B_n and F_n no longer represent degenerate excitations. However B_n and $F_{n_1}F_{n-n_1-1}$ do, for every value of n_1. Thus, as before, the degeneracies can be counted using Young diagrams. For a given excitation of energy n one again forms all the Young diagrams corresponding to the partitions of n. These are simply all the zero Fermion number excitations. One can then replace each column of the Young diagram (say of length m) with two crossed columns of lengths m_1 and m_2 satisfying

$$m_1 + m_2 = m - 1. \tag{3.29}$$

The new columns have to be added in a way such that the new diagram is still a legal Young diagram. Each such replacement is equivalent to replacing three Bosonic excitations with two Fermionic ones. Carrying this process out for all the columns of the diagrams generates for us all the F type excitations that are degenerate to a state of a given energy. In the process of generating Fermionic excitations, one also needs to exercise the constraint that there cannot be two crossed columns of the same length in a given Young diagram.

The effect of replacing Bosonic excitations by Fermionic ones of length 1 on a particular young diagram is illustrated in the following diagram.



In the usual analysis of Calogero systems with a finite number (N) of particles, one imposes a non-perturbative cutoff on the depth of the columns of the Young diagrams. Namely, the columns are not allowed to have more than N boxes. However, that would correspond to the finite N matrix model, for which the states that we picked are no longer protected. The large N limit, translates, in the language of the Young diagrams to lifting the non-perturbative cutoff on the depth of the columns. Looked at in another way, imposing the BPS condition at the level of the Calogero system is equivalent to lifting the cutoff on the depth of the columns.

4. Remnants of Yangian symmetries and loop algebras

In this section, we shall focus on the realization of Yangian symmetries and non-local conservation laws in the super-Calogero system.

The Calogero model is nothing but the gauge fixed form of the dilatation operator in a particular sector of BPS operators. If one goes beyond the BPS sectors, one would of course have to incorporate the perturbative corrections to the dilatation generator. At finite values of N this is hard to accomplish, however from the detailed studies of operator mixing in the gauge theory in the recent past, the first few perturbative corrections to the large N limit of the dilatation operator are known in rather explicit forms, at least in some small sectors of operator mixing. For example, in the $su(2|3)$ sector discussed earlier, the planar dilatation operator is known up to the third order in the 't Hooft coupling [11]. It has also been shown that the dilatation operator can be realized as an integrable quantum spin chain up to this order in perturbation theory [11, 42]. One point of view on the integrability of the spin chain relates the integrability to the existence of Hopf algebraic symmetries: the integrability being simply the manifestation of such large hidden symmetries. For more detailed studies of the Yangian for the gauge theory we shall refer to [44, 45, 43]. The existence of Yangian symmetries, apart from providing key insights into the algebraic structures that are responsible for the integrability of the spin chain, are also crucial from the point of view of the AdS/CFT correspondence as the string sigma model has been known to possess this very same symmetry at the classical level [47, 48, 46, 50, 49]. To the extent that the spectrum of anomalous dimensions of the gauge theory and those closed string excitations agree (for instance in the BMN limit) it has been possible to relate the Yangian symmetries on the gauge theory and the gravity sides. For the specific case of

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studies of integrable structures in the the $su(1|1)$ sector of the AdS/CFT correspondence we shall refer to [45, 51, 52]

However, it is not clear at the moment whether or not these novel symmetries survive the low energy supergravity limit. One can however use the gauge theory as a probe to investigate this problem. Since results from the BPS sectors of the gauge theory can be extrapolated to the supergravity limit one can investigate the role of the Yangian symmetries of the dilatation operator when it is restricted to the BPS states and try and understand how these symmetries manifest themselves in the supergravity limit. With this motivation in mind we can probe the structure of Yangian symmetries of the super Calogero model studied so far.

The Yangian charges and the conserved integrals of motion of the Calogero model are generated by the matrix elements of the transfer matrix, which is a 2×2 matrix for the $su(1|1)$ model, each matrix element of which is an operator in the Hilbert space of the Calogero model [53].[3] The transfer matrix has a free parameter, namely the spectral parameter u and the standard expansion around an infinite vale of the spectral parameter reads as

$$T^{ab} = I\delta^{ab} + \sum_{n=1}^{\infty} \frac{1}{u^n} T^{ab}_{n-1}. \tag{4.1}$$

In the above expression $S^{ab}(j)$ are the $su(1|1)$ generators at the j th lattice site,

$$\begin{aligned} S^{11}(j) = \mathcal{A}_j \mathcal{A}_j^\dagger, S^{22}(j) = \mathcal{A}_j^\dagger \mathcal{A}_j \\ S^{12}(j) = \mathcal{A}_j, S^{21}(j) = \mathcal{A}_j^\dagger. \end{aligned} \tag{4.2}$$

$$T^{ab}_n = \sum_{j,k} S^{ab}(j)(L^n)_{j,k}, \tag{4.3}$$

where L^n is the n th power of the $N \times N$ Lax matrix

$$L_{j,k} = \delta_{j,k}\left(\frac{\partial}{\partial x_j} + x_j\right) + \hbar(1-\delta_{j,k})\omega_{j,k}\Pi_{j,k} \tag{4.4}$$

and

$$\omega_{j,k} = \frac{e^{-\frac{\hbar}{2}(x_i - x_j)}}{\sinh \frac{\hbar}{2}(x_i - x_j)} \tag{4.5}$$

We have chosen to incorporate a free parameter, which we suggestively denote by $\hbar$ in the above analysis to illustrate the contraction of the Yangian algebra to the loop algebra in a transparent way. We have also chosen an inverse hyperbolic fall off of the inter-particle potential in the Lax operator rather than the $1/(x_i - x_j)$ fall off for the same purpose. The basic idea being to start with the hyperbolic case, which contains the rational and the trigonometric Calogero models as special cases and recover the underlying symmetry of the rational case as a particular limit.

[3]There is a large literature on the role of Yangian symmetries and quantum spin chains. Of particular relevance to the present problem are [54–58].

The transfer matrix satisfies the quadratic Yang-Baxter algebra.

$$[T_s^{ab}, T_{p+1}^{cd}]_\pm - [T_{p+1}^{ab}, T_s^{cd}]_\pm = \hbar(-1)^{\epsilon(c)\epsilon(a)+\epsilon(c)\epsilon(b)+\epsilon(b)\epsilon(a)} \left(T_p^{cb} T_s^{ad} - T_s^{cb} T_p^{ad}\right). \tag{4.6}$$

In the above equation ϵ denotes the grade $\epsilon(1) = 0, \epsilon(1) = 1$. It is important to note that the non-linearity of the Yang-Baxter algebra (the r.h.s of the above equation) is proportional to $\hbar$. The Yang-Baxter algebra also implies

$$\sum_{i,j} [T_m^{ii}, T_n^{jj}] = 0 \tag{4.7}$$

i.e the trace of the transfer matrix is the generating function for the conserved charges which are in involution. As a matter of fact, if one denoted these charges by $H_n = \mathrm{tr} T^n$, then one can show that the Hamiltonian, up to the addition of constant terms is nothing but T^2, which for the Hyperbolic case takes on the following form.

$$H = \frac{1}{2} \sum_{j,k} (-\partial_j^2 + x_j^2 + \mathcal{A}^\dagger(j)\mathcal{A}(j) + \hbar \Pi_{j,k} \partial_j \omega_{j,k} + \hbar^2 \omega_{j,k} \omega_{k,j}) \tag{4.8}$$

For the limit of interest to us, $\hbar \to 0$ we recover

$$\omega_{j,k} = \frac{1}{x_j - x_k} \tag{4.9}$$

with the Hamiltonian above becoming the super-Calogero Hamiltonian:

$$H \to \sum_i \left(-\frac{1}{2} \frac{\partial}{\partial x_i^2} + 2\mathcal{A}^{\dagger i} \mathcal{A}_i + \frac{1}{2} x_i^2 \right) + \frac{1}{2} \sum_{i \neq j} \left(\frac{1 - \Pi_{i,j}}{(x_i - x_j)^2} \right). \tag{4.10}$$

As is obvious from the above construction, in this 'clasical' limit, the Yangian algebra degenerates into the loop algebra:

$$[T_s^{ab}, T_{p+1}^{cd}]_\pm - [T_{p+1}^{ab}, T_s^{cd}]_\pm = 0, \tag{4.11}$$

which can be written, upon using the above relations recursively as:

$$[T_s^{ab}, T_p^{cd}]_\pm = \delta_{b,c} T_{p+s}^{ad} - (-1)^{(\epsilon(a)+\epsilon(b))(\epsilon(c)+\epsilon(d)} \delta_{a,d} T_{p+s}^{cb} \tag{4.12}$$

Hence, the integrable structure in the dynamics of the $su(1|1)$ BPS operators appears to arise from the loop algebra of $su(1|1)$. One might have anticipated this from the fact that the dynamics of the BPS sectors of the gauge theory can be extrapolated to the supergravity regime and the supergravity can be regarded as a classical limit of the string theory. On the other hand the loop algebra is also a classical limit of the Yangian algebra, which appears to be a symmetry of the dual string theory. The discussion above indicates, through an explicit construction, that these two notions of classical limits are compatible with each other.

Furthermore, we can also see that the supersymmetry generators are contained in the loop algebra. As a matter of fact it is easy to see that:

$$T_1^{21} = Q, T_1^{12} = Q^\dagger \tag{4.13}$$

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and that:

$$H = [T_1^{21}, T_1^{12}]_+ \tag{4.14}$$

The higher (odd) elements of the loop algebra simply act as the supersymmetry generators for the higher conserved charges of the system:

$$H_{n+m} = [T_n^{12}, T_m^{21}]_+ \tag{4.15}$$

We thus see that the loop algebra and the supersymmetry of the particle mechanics fit together in a natural way.

The Calogero model Hamiltonian is, of course, not the dilatation operator, as the two differ by a term proportional to the fermion number operator

$$H' = H + \frac{1}{2}\mathcal{A}_i^\dagger \mathcal{A}_i \tag{4.16}$$

However, just as we were able to recover the degeneracies of the dilatation operator from those of the Calogero model, it is possible to use the transfer matrix of the Calogero system to construct the integrals of motion for the dilatation operator. The construction is extremely simple. The Fermion number operator does not commute with the supersymmetry generators, and in general, with the odd elements of the Yangian algebra T_n^{12} and T_m^{21}. However, it does commute with the generators of even grade. Thus we have

$$[H', T_n^{ii}] = 0 \quad \forall i \tag{4.17}$$

Moreover, from the loop algebra it is clear that

$$[T_m^{11}, T_n^{11}] = [T_m^{11}, T_n^{22}] = [T_m^{22}, T_n^{22}] = 0 \quad \forall m, n \tag{4.18}$$

Thus we have as many conserved charges in involution for the dilatation operator as there are degrees of freedom; namely $2N$. Thus we recover the integrability of the dilatation operator from the underlying loop algebraic symmetry of the super Calogero model.

5. Spectrum of the Euler-Calogero system

We shall now revert back to the general $su(2|3)$ Hamiltonian given in (2.20). Integrability of the particle mechanics model presented above derives from the fact that it is nothing but a sum of decoupled (matrix) oscillators in disguise. Such matrix models are obviously integrable, indeed even for finite values of N, and they continue to be integrable in the large N limit. Apart from the explicit solutions to the equations of motion of these matrix models, integrability also manifests in the existence of a large number (infinite in the large N limit) of conserved quantities. It is worthwhile to understand the integrability of the many-body system in some detail. To do that let us begin by writing the Hamiltonian in the special basis in which X^1 is diagonal:

$$H = \sum_\alpha (a^{\dagger\alpha} a_\alpha). \tag{5.1}$$

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It is understood that

$$(a_1)^i_j = \left(x_i + \frac{\partial}{\partial x_i}\right)\delta^i_j + \frac{(1-\delta^i_j)\mathcal{L}^i_j}{x_i - x_j}$$
$$(a^{\dagger 1})^i_j = \left(x_i - \frac{\partial}{\partial x_i}\right)\delta^i_j + \frac{(1-\delta^i_j)\mathcal{L}^i_j}{x_i - x_j} \tag{5.2}$$

and the other oscillators $a^{\dagger\alpha}, a_\alpha$ $(\alpha \neq 1)$ are simply the remaining degrees of freedom for the gauged fixed matrix model i.e they are the U(N) rotated oscillators. Translating the original matrix model equations of motion to this special basis one can see that

$$\dot{a}^{\dagger\alpha} = a^{\dagger\alpha} + [a^{\dagger\alpha}, g]$$
$$\dot{a}_\alpha = -a_\alpha + [a_\alpha, g] \tag{5.3}$$

where the commutator on the r.h.s is the matrix commutator and

$$g^i_j = \frac{(1-\delta^i_j)\mathcal{L}^i_j}{x_i - x_j} \tag{5.4}$$

It is now a straightforward exercise to show that operators

$$(O)^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n} = \mathrm{tr}\left(a^{\dagger\alpha_1}\cdots a^{\dagger\alpha_m} a_{\beta_1}\cdots a_{\beta_n}\right) \tag{5.5}$$

evolve according to

$$\dot{\mathcal{O}}^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n} = (m-n)(O)^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_n} \tag{5.6}$$

This obviously implies that $(O)^{\alpha_1\cdots\alpha_n}_{\beta_1\cdots\beta_n}$ are all integrals of motion for every n and that the states

$$|\{\alpha_{i_1}\cdots\alpha_{i_m}\}\rangle = \mathrm{tr} a^{\dagger\alpha_1}\cdots a^{\dagger\alpha_m}|0\rangle \tag{5.7}$$

are exact eigenstates of H with energy m. Thus, quite like the super-Calogero model the degeneracies can once again be counted by the use of Young diagrams. In the zero Fermion number sector, $\mathcal{L}^i_j = 0$, and hence, all the states with a given energy n can be labeled by Young diagrams corresponding to the partitions of n. But unlike the Calogero model, one now has four types of impurities, two Bosonic and two Fermionic. Just as we introduced diagrams with crossed columns in the Calogero case, here we have to distinguish between the various impurities, and hence it is useful to think of the columns being colored by four colors corresponding to the impurities. Thus the additional degeneracies are generated by replacing the columns of the Young diagrams of the zero impurity number sector with colored columns one at a time. We also have to keep in mind that when we add columns corresponding to the Bosonic impurities, we trade a column of the original free Fermion Young diagram for a colored column of the same length. However, as in the case of the $su(1|1)$ sector, when it comes to inserting Fermionic impurities, one has to replace the columns of the free Fermion Young diagram (say of size n) with two columns, one of lengths n_1 and n_2 satisfying

$$n_1 + n_2 = n - 1. \tag{5.8}$$

Furthermore, we need to make sure that the there is at the most one Fermionic column of a given color and length.

This construction counts all the degeneracies between states that have at the most one impurity inside a single trace in the original matrix model picture. These are half BPS states, although not all states of the matrix model are. The degeneracies between states of zero impurity number and impurity number greater than one are not accounted for by the above construction. For example, the above construction does not count the degeneracy between the states

$$\mathrm{tr}(a^{1\dagger})^{m+n+2}|0\rangle \quad \text{and} \quad \mathrm{tr}\left((a^{1\dagger})^m a^{2\dagger}(a^{1\dagger})^n a^{2\dagger}\right)|0\rangle. \tag{5.9}$$

The second state above is non-BPS. Thus, we have been able to utilize the integrability of the Euler-Calogero model to enumerate all the BPS states, that are charged under $su(2|3)$, formed out of inserting a single impurity field inside a matrix trace.

A natural question that arises is how one may describe BPS excitations involving several impurity fields within the open string picture. The answer to that is not hard to see. One needs to write the supercharges for the full $su(2|3)$ sector in the basis in the which X^1 is diagonal. Since the BPS states are all generated by the action of the supersymmetry generators, all one needs to do is write the super charges in this basis and generate all the BPS states by their repeated action. The supercharge of interest to us is the one that replaces a scalar impurity field by a Fermionic one and it can be written as 2×3 matrix, with matrix elements

$$Q^I_i = \mathrm{tr}(\Psi^{\dagger I} a_i) \tag{5.10}$$

which in the basis of interest takes the form

$$Q^\alpha_\beta = \mathrm{tr}(a^{\dagger\alpha} a_\beta), \quad \alpha = 4,5,\ \beta = 1,2,3. \tag{5.11}$$

Needless to say, in this second form it is implied that the oscillators are the ones in the $\mathrm{U}(N)$ rotated basis (5.2).

The construction described previously enumerates all the BPS states formed out of single action of the supercharge. The rest can be similarly generated by repeating the action of the supersymmetry generator given above. This will pick out all the BPS states contained in the complete set of states of the Euler-Calogero model.

6. Discussion and future directions

The general connection between the dilatation operator and Calogero systems can lead to several interesting avenues of investigation that were not addressed in the present work. We list some of these possibilities below.

1. From the point of view of integrable systems, it would be extremely interesting to study the integrability of the Euler-Calogero system in greater detail. In the present work, we presented enough of an understanding of its integrability to understand its spectrum and the associated degeneracies. Gaining an understanding of the underlying Yang-Baxter algebra for the quantum Euler-Calogero system would clarify the

role of Yangian type symmetries for this system. Such an analysis should be possible, as the classical r matrix for the Euler-Calogero system, which curiously enough is a dynamical 'r' matrix has been found in [59]. Of particular interest would be the a systematic understanding of the dynamical models and integrable structures that arise when one considers BPS operators that involve more than a single impurity field inside a trace.

2. In the paper we showed that the rational super-Calogero model can be regarded as the simplest non-trivial generalization of the theory of free Fermions when it comes to understanding protected operators of the gauge theory. Just like the theory of free Fermions, it was shown that one can have two equivalent description of the states of this theory, which we regarded as an open/closed duality. Clearly it would be extremely desirable to have a world sheet interpretation of the super-Calogero system, along the lines of the description provided in [12] for the free Fermion system. It is not hard to envisage what the world sheet string theory would be. The string dual of the free Fermion system was found by taking valuable clues from string theory in two dimensions and analytically continuing the string dual of the $C = 1$ matrix model to the case of the 'right-side-up' harmonic oscillator. To take a similar clue for the string dual of the Calogero model, we shall have to look at the world-sheet description of strings in AdS_2. This particular string theory was analyzed recently in [32] and a connection to Calogero systems was also made in the same paper. It seems plausible that this very theory is the string dual of the $su(1|1)$ BPS sector of $\mathcal{N}$ =4 SYM discussed earlier in this paper. We hope to report on this possibility in the near future.

3. Clearly, the tree level dilatation operator can be written as an Euler-Claogero system even if the states in question do not correspond to BPS operators of the gauge theory. Hence the Euler-Calogero system provides us with a starting point for understanding non-BPS excitations. It would indeed be extremely interesting to understand how this framework of the Euler-Calogero model changes once the higher loop corrections to the dilatation operator are considered. Recently it has been shown that it is possible to obtain the all-loop BMN formula by doing a one loop computation around a carefully chosen vacuum of the dilatation operator [22, 23]. This point of view can be easily incorporated within the formalism developed in the present paper. We hope to report on the connection of Euler-Calogero type of dynamical systems and non-BPS corrections to the supergravity spectrum in the near future as well.

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